

A comparison Between Synchronized Permutation and Two-way ANOVA Test Statistics

Confronto tra tests con permutazioni sincronizzate e tests ANOVA a due vie

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Riassunto: In questo lavoro presentiamo i test di permutazione per piani fattoriali replicati $I \times J$ (Pesarin, 2001; Salmaso 2003). La soluzione proposta non necessita la specificazione della distribuzione degli errori sperimentali. Se può essere assunta la normalità degli errori, i test con permutazioni sincronizzate danno risultati sostanzialmente equivalenti a quelli ottenuti con le statistiche dell'ANOVA a due vie. Inoltre, mentre le statistiche ANOVA risultano correlate positivamente, le statistiche test con permutazioni sincronizzate risultano incorrelate, almeno per quanto riguarda i fattori principali. In questo lavoro riportiamo in forma generale il grado di correlazione tra le statistiche ANOVA per i fattori principali come funzione del numero di livelli di ciascun fattore, del numero di repliche e della eventuale non centralità.

Keywords: synchronized permutations, noncentral F distribution.

1. Introduction to synchronized permutations

In a $I \times J$ replicated factorial design, data are assumed to follow the linear model:

$$y_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, \dots, I \\ j = 1, \dots, J \\ k = 1, \dots, n \end{cases} \quad (1)$$

where y_{ijk} are experimental responses, μ is the overall mean, α_i, β_j are main factor effects, $\alpha\beta_{ij}$ are interaction effects, ε_{ijk} are exchangeable experimental errors, with zero mean and variance σ^2 , from an unknown distribution P , and n is the number of replicates in each cell. We also assume the usual side-conditions $\sum_i \alpha_i = 0, \sum_j \beta_j = 0, \sum_i \alpha\beta_{ij} = 0 \forall j, \sum_j \alpha\beta_{ij} = 0 \forall i$. The hypotheses of interest are $H_{0A} : \alpha_i = 0 \forall i$ against $H_{1A} : \exists i : \alpha_i \neq 0$, $H_{0B} : \beta_j = 0 \forall j$ against $H_{1B} : \exists j : \beta_j \neq 0$ and $H_{0AB} : \alpha\beta_{ij} = 0 \forall i, j$ against $H_{1AB} : \exists i, j : \alpha\beta_{ij} \neq 0$. Since the observations from different cells have different means, they are not exchangeable. To perform nonparametric testing on main factor effects, we need to introduce a restricted kind of randomization. Suppose we want to test for H_{0A} and that ν units are randomly exchanged between blocks $A_i B_j$ and $A_s B_j$, $j = 1, \dots, J$. Let us consider the permutation structure of an intermediate statistics to separately compare levels i and s of factor A at level j of factor B:

$${}^a T_{is|j}^* = \sum_k y_{ijk}^* - \sum_k y_{sjk}^* = (n - 2\nu)[\alpha_i - \alpha_s + \alpha\beta_{ij} - \alpha\beta_{sj}] + \bar{\varepsilon}_{is|j}^* \quad (2)$$

where the symbol “*” denotes that we have obtained it after exchanging ν units between cells. Then, due to the side conditions, the test statistic:

$${}^aT_A^* = \sum_{i < s} [\sum_j {}^aT_{is|j}^*]^2, \quad (3)$$

only depends on levels of factor A and on exchangeable errors ($\bar{\varepsilon}_{is|j}^*$), hence it is a separate exact test on factor A. ${}^aT_A^*$ is a sum of all possible pairs of cells of the intermediate statistics ${}^aT_{is|j}^*$, $j = 1, \dots, J$. In a similar way, define ${}^bT_{jh|i}^* = \sum_k y_{ijk}^* - \sum_k y_{ihk}^*$ the intermediate statistic for comparing levels j and h of factor B at level i of factor A. Then the test statistic:

$${}^bT_B^* = \sum_{j < h} [\sum_i {}^bT_{jh|i}^*]^2, \quad (4)$$

only depends on levels of factor B and on exchangeable errors, hence it is a separate exact test on factor B. We can also define two test statistic for the interaction:

$${}^aT_{AB}^* = \sum_{i < s} \sum_{j < h} [{}^aT_{is|j}^* - {}^aT_{is|h}^*]^2 \quad \text{and} \quad {}^bT_{AB}^* = \sum_{j < h} \sum_{i < s} [{}^bT_{jh|i}^* - {}^bT_{jh|s}^*]^2. \quad (5)$$

Note that ${}^aT_{AB}^*$ is obtained from synchronized permutations involving the row factor A, whereas ${}^bT_{AB}^*$ is obtained from permutations involving the column factor B. Each statistic for interaction only depends on interaction levels and on exchangeable errors, hence it is a separate exact test on interaction AB. Wishing only one test, a nonparametric combination may follow.

2. Correlation between F_A and F_B , uncorrelation between ${}^aT_A^*$ and ${}^bT_B^*$

It is known that the two-way ANOVA test statistics are ratios between independent quadratic forms. In particular, the main factor test statistics are define as:

$$F_A = \frac{SS_A/(I-1)}{SS_\varepsilon/IJ(n-1)} \quad \text{and} \quad F_B = \frac{SS_B/(J-1)}{SS_\varepsilon/IJ(n-1)},$$

where $SS_A/(I-1)$ and $SS_B/(J-1)$ are unbiased estimates of σ^2 only if the related null hypothesis is true, whereas $SS_\varepsilon/IJ(n-1)$ is always an unbiased estimate of σ^2 . Since both statistics have the same denominator, they are positively correlated. In this section we obtain the correlation between F_A and F_B as a function of I, J, n , and noncentrality parameters α and β . We want to evaluate:

$$\rho(F_A, F_B) = \frac{COV(F_A, F_B)}{\sqrt{Var(F_A)}\sqrt{Var(F_B)}},$$

where F_A and F_B are noncentral F if the corresponding alternative is true. Given that SS_A, SS_B and SS_ε are independent χ^2 variables, we have:

$$\begin{aligned} COV[F_A, F_B] &= E[F_A \cdot F_B] - E[F_A]E[F_B] \\ &= E[SS_A/(I-1)]E[SS_B/(J-1)]E[I^2J^2(n-1)^2SS_\varepsilon^{-2}] + \\ &\quad - E[SS_A/(I-1)]E[B/(J-1)]E[IJ(n-1)SS_\varepsilon^{-1}]^2 \\ &= E[MS_A]E[MS_B]Var[IJ(n-1)SS_\varepsilon^{-1}] > 0. \end{aligned}$$

Being a χ^2 random variable, SS_ϵ belongs to the Gamma family, so it is easy to obtain $Var[SS_\epsilon^{-1}]$. More precisely, $SS_\epsilon \sim Ga(ab(n-1)/2, 1/2\sigma^2)$, hence:

$$Var[IJ(n-1)SS_\epsilon^{-1}] = \frac{2I^2J^2(n-1)^2}{[IJ(n-1)-2]^2[IJ(n-1)-4]\sigma^4}$$

and:

$$\begin{aligned} Var(F_A) &= \frac{2I^2J^2(n-1)^2}{(I-1)^2} \times \frac{\alpha^2 + [(I-1) + 2\alpha][I-1 + IJ(n-1) - 2]}{[IJ(n-1)-2]^2[IJ(n-1)-4]} \\ Var(F_B) &= \frac{2I^2J^2(n-1)^2}{(J-1)^2} \times \frac{\beta^2 + [(J-1) + 2\beta][J-1 + IJ(n-1) - 2]}{[IJ(n-1)-2]^2[IJ(n-1)-4]}. \end{aligned}$$

Hence:

$$\begin{aligned} \rho(F_A, F_B) &= \frac{E[MS_A]E[MS_B]Var[IJ(n-1)SS_\epsilon^{-1}]}{[Var(F_A)Var(F_B)]^{1/2}} \\ &= \frac{[(I-1) + \alpha][(J-1) + \beta]}{\sqrt{\tilde{\sigma}^2(F_A)\tilde{\sigma}^2(F_B)}}, \end{aligned} \quad (6)$$

where:

$$\begin{aligned} \tilde{\sigma}^2(F_A) &= \alpha^2 + [(I-1) + 2\alpha][(I-1) + ab(n-1) - 2], & \alpha &= Jn \sum_i \alpha_i^2 / \sigma^2; \\ \tilde{\sigma}^2(F_B) &= \beta^2 + [(J-1) + 2\beta][(J-1) + ab(n-1) - 2], & \beta &= In \sum_j \beta_j^2 / \sigma^2. \end{aligned}$$

From (6), we see that: $\lim_{\alpha, \beta \rightarrow +\infty} \rho(F_A, F_B) = 1$.

Let us consider the synchronized permutation test statistics for main factors now. The tests on main factors are defined on different permutation spaces. In particular, ${}^aT_A^*$ is obtained by exchanging units between pairs of rows and within columns, ${}^bT_B^*$ is obtained by exchanging units between pairs of columns and within rows. Hence, without loss of generality, consider the case when we are testing for $H_{0A} : \alpha_i = 0 \forall i$. This hypothesis allows us to exchange units between rows and within columns. The test statistic ${}^aT_A^*$ is a random variable and its probability function $P({}^aT_A^*)$ is defined on the support of ${}^aT_A^*$. Units are exchanged within columns, so the total of each column is constant. Let $Y_{.j}, j = 1, \dots, J$ be the total of column j . Then:

$$\sum_i {}^bT_{jh|i}^* = \sum_i [\sum_k y_{ijk}^* - \sum_k y_{ihk}^*] = \sum_i \sum_k y_{ijk}^* - \sum_i \sum_k y_{ihk}^* = Y_{.j} - Y_{.h}$$

hence:

$${}^bT_B^* = \sum_{j < h} [\sum_i T_{jh|i}^*]^2 \equiv \sum_{j < h} (Y_{.j} - Y_{.h})^2 = {}^bT_B$$

where bT_B is the observed value of the test statistic for factor B. Since ${}^bT_B^*$ is constant, $\rho({}^aT_A^*, {}^bT_B^*) = 0$. The same can be done w.r.t. factor A when testing for H_{0B} .

There are two ways to obtain a synchronized permutation: exchanging units with the same original position in each pair of cells (*Constrained Synchronized Permutations*, CSPs), or exchanging units in each pair of cell without regarding their original position

(Unconstrained Synchronized Permutations, USPs). Figures 1 and 2 show a comparison between two-way ANOVA and CSPs test statistics. The graphs are scatterplots of the main factor p-values obtained with each test. The simulation is based on 1000 resamplings from a Normal distribution in accordance with model 1, with $n = 3$. In Figure 1 no active effects are present, whereas in Figure 2 we set $\alpha_1 = -\alpha_2 = 10\sigma$, $\beta_1 = -\beta_2 = 5\sigma$.

Figure 1: Left: Scatterplot of $p(^aT_A^*)$ vs. $p(^bT_B^*)$. Right: Scatterplot of $p(F_A)$ vs. $p(F_B)$.

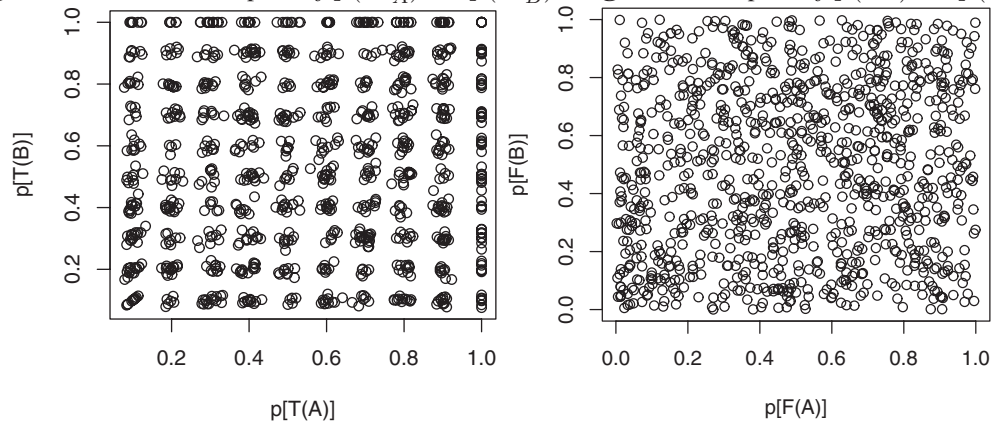
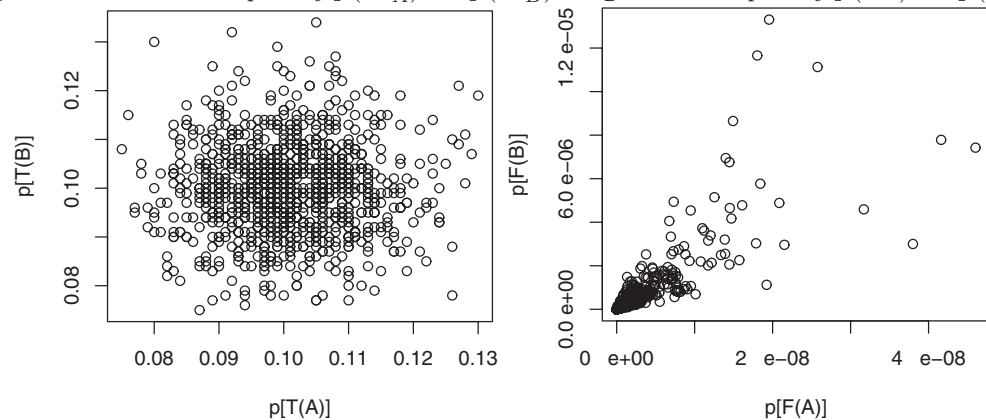


Figure 2: Left: Scatterplot of $p(^aT_A^*)$ vs. $p(^bT_B^*)$. Right: Scatterplot of $p(F_A)$ vs. $p(F_B)$.



Note how, under the null hypothesis (Figure 1), dots are randomly spread in the interval $[0, 1]^2$ either by applying the CSP test or the F test. This does not happen when α and β are not null: the correlation between ANOVA tests is evident (Figure 2), whereas dots obtained from the CSP test are still randomly spread around the minimum achievable p-value (here, the CSP attainable p-values are multiple of $1/10$).

References

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